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Phase Transition in Ferromagnetic Ising Models with Non-uniform External Magnetic Fields

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Abstract In this article we study the phase transition phenomenon for the Ising model under the action of a non-uniform external magnetic field. We show that the Ising model on the hypercubic lattice with a summable magnetic field has a first-order phase transition and, for any positive (resp. negative) and bounded magnetic field, the model does not present the phase transition phenomenon whenever $\liminf h_i > 0$, where $\mathbf{h} = (h_i)_{i\in\mathbb{Z}^d}$ is the external magnetic field.

Keywords Ising model · Phase transition · Phase uniqueness · Non-uniform magnetic field

1 Introduction

The Lee-Yang theorem [10] is one of the most revisited results in Statistical Mechanics [11–13], especially because of its application to the study of phase transition phenomena. One consequence of this theorem is that for any nonzero *uniform* magnetic field, i.e. $\mathbf{h} = (h_i)_{i\in\mathbb{Z}^d}$, $h_i = h \in \mathbb{R}\backslash\{0\}$ for all $i \in \mathbb{Z}^d$ and $\beta = 1/kT$, the ferromagnetic ($J > 0$) Ising model on $\mathbb{Z}^d$ has an unique Gibbs measure in the thermodynamic limit, independently of the boundary conditions.

In this paper we consider more general models where the magnetic field $\mathbf{h}$ is not supposed to be uniform. For such models the Lee-Yang Theorem is still valid, and a natural question is to ask if for these models the Lee-Yang Theorem still implies the absence of phase transition. The question of the uniqueness of the Gibbs measure in a non-uniform positive magnetic

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field it was considered by Georgii [5] and Fontes and Neves [3]. They considered the model with a non-negative random field of positive mean, and proved that for all temperatures that there exists a unique Gibbs state. Here, in the Theorem 4 we prove uniqueness of the Gibbs states at all temperatures for all models for which $\liminf h_i > 0$ (no average assumption is required for the fields h_i), showing that the Lee-Yang Theorem, in this case, still implies the absence of phase transition. Although we present a proof for Theorem 4 in the hypercubic lattice, the same argument works for any connected amenable quasi-transitive graph with bounded degree.

It is well known that the Ising model with positive magnetic field can present phase transition depending on the graph structure of the model. Jonasson and Steif in [7] showed that the Lee-Yang Theorem does not imply the absence of a first-order phase transition for this model in any nonamenable graph. Basuev in [1] obtained the same result for a class of amenable but not quasi-transitive graphs. In this paper, assuming that the magnetic field decays to zero, we obtain the same result for hypercubic lattices, which are examples of amenable and quasi-transitive graphs. In other words, even with the magnetic field taking positive values at all sites of the hypercubic lattice, we prove that the Ising model can present a first-order phase transition.

2 Preliminaries and Main Results

Consider the distance between x and y on $\mathbb{Z}^d$ given by $\|x-y\| = \sum_{i=1}^{d} |x_i - y_i|$ and, for any finite $\Lambda \subset \mathbb{Z}^d$, denote by $\partial\Lambda$ the set of sites in $\mathbb{Z}^d$ whose distance to Λ is equal to 1. The energy in Λ of each configuration $\sigma \in \Omega \equiv \{-1,+1\}^{\mathbb{Z}^d}$ satisfying the *boundary condition* $\omega \in \Omega$ in Λ (ω b.c.), that is, $\sigma_i = \omega_i\ \forall i \in \mathbb{Z}^d \backslash \Lambda$, is given by the Hamiltonian

$$H_{\Lambda}^{\omega}(\sigma) = -\frac{J}{2}\sum_{\langle i,j\rangle}\sigma_i\sigma_j - \sum_{i\in\Lambda} h_i\sigma_i, \tag{1}$$

where $\langle i, j\rangle$ denotes the set of ordered pairs in $\Lambda \cup \partial\Lambda$ of nearest neighbors. For $J > 0$ and $\mathbf{h} = (h_i)_{i\in\mathbb{Z}^d}$ the Hamiltonian defines a ferromagnetic Ising model with external field $\mathbf{h}$.

The *Gibbs measure in* Λ *with* ω *b.c.* is the probability measure on $(\Omega, \mathcal{F})$, where $\mathcal{F}$ is the sigma-algebra generated by the cylinder sets given by

$$\mu_{\Lambda}^{\beta,\mathbf{h},\omega}(\sigma) = \frac{e^{-\beta H_{\Lambda}^{\omega}(\sigma)}}{Z_{\Lambda}^{\omega}} \tag{2}$$

if σ satisfies the ω b.c. and zero otherwise. The normalization factor is the standard *partition function*

$$Z_{\Lambda}^{\omega} = \sum_{\sigma} e^{-\beta H_{\Lambda}^{\omega}(\sigma)} \tag{3}$$

with the sum over all configurations σ satisfying the ω b.c.

We denote by $\mathcal{G}_{\beta,\mathbf{h}}$ the set of *Gibbs measures* given by the closed convex hull of the set of weak limits:

$$\mu^{\beta,\mathbf{h},\omega} = w - \lim_{\Lambda_n \nearrow \mathbb{Z}^d} \mu_{\Lambda_n}^{\beta,\mathbf{h},\omega}, \tag{4}$$

where $\Lambda_n \subset \Lambda_{n+1}$ and ω runs over all boundary conditions.

We say that there is a first-order *phase transition* when the set $\mathcal{G}_{\beta,\mathbf{h}}$ contains more than one measure, see [2, 6]. In the case of a non-zero uniform field, the Lee-Yang theorem can be used to prove that the analyticity of the pressure with respect to the parameter h is equivalent to saying that the system has no phase transition, see [16].

We write $\mathbf{h} \in \ell^1(\mathbb{Z}^d)$ when $\|\mathbf{h}\|_1 = \sum_{i\in\mathbb{Z}^d} |h_i| < \infty$ and $\mathbf{h} \in \ell^\infty(\mathbb{Z}^d)$ when $\mathbf{h}$ is bounded, that is, $\sup_{k\in\mathbb{Z}^d} |h_k| < \infty$.

The paper is organized as follows: Our main result is presented in Sect. 3, roughly speaking it states that, if the external field $\mathbf{h}$ is summable on $\mathbb{Z}^d$, that is, $\mathbf{h} \in \ell^1(\mathbb{Z}^d)$, then the model has a first-order phase transition. This result is obtained by a Peierls-type argument, based on contours already used in [15, 17] and a cancelation argument for clusters of contours with opposite signs.

In Sect. 4, we address the question of non-summable $\mathbf{h}$. We show that, if we have $\liminf h_i > 0$ (similarly for $\limsup h_i < 0$), then the system has no first-order phase transition. We remark that although this fact is pretty obvious, from the physical point of view, the arguments we found in the literature are based upon translation-invariance [9]. For a general $\ell^\infty(\mathbb{Z}^d)$ magnetic field it is not clear that we can use those techniques. By controlling the Radon-Nikodyn derivative we show how to deal with the non-invariant case.

3 The Expansion in Contours and the Phase Transition

The arguments in this section can be generalized for higher dimensions, we take dimension 2 for simplicity. We will describe roughly the *contours* in $\mathbb{Z}^2$, for details the readers can see [15, 17]. This approach uses a bijection between finite collections of contours in $\mathbb{Z}^{2*} = \mathbb{Z}^2 + (1/2, 1/2)$ and configurations with some fixed boundary condition. Without loss of generality, we always suppose that the finite set Λ is a square. This fact helps us to assure that, if we have a finite number of contours in the dual set of Λ and already fixed a boundary condition, then there is a configuration associated to these contours, see Lemma 2.1.2 [17].

Theorem 1 *If the magnetic field* $\mathbf{h}$ *in the Hamiltonian in* (1) *belongs to* $\ell_1(\mathbb{Z}^2)$, *then the model presents a phase transition when* $J > 3\|\mathbf{h}\|_1$.

Proof let $\Lambda \subset \mathbb{Z}^2$ be a finite set and suppose initially that $h_i \geq 0$ $\forall i \in \mathbb{Z}^2$. It follows from the second Griffiths inequality that

$$\langle\sigma_i\rangle_\Lambda^{\beta,\mathbf{h},+} \geq \langle\sigma_i\rangle_\Lambda^{\beta,\mathbf{0},+} = 1 - 2\mu_\Lambda^{\beta,\mathbf{0},+}(\{\sigma_i = -1\}),$$

where $\mathbf{0}$ is the null magnetic field. Using this lower bound and the following standard inequality (see [14] page 170 for details)

$$\mu_\Lambda^{\beta,\mathbf{0},+}(\{\sigma_i = -1\}) \leq c(\beta) := \sum_{n=4}^{\infty}(2n+3)3^{n-1}e^{-2\beta J n},$$

since clearly $c(\beta) \to 0$, uniformly in Λ when $\beta \to \infty$, we get that

$$\lim_{\beta\to\infty} \langle\sigma_i\rangle^{\beta,\mathbf{h},+} = 1. \tag{5}$$

The next step is to show that $\lim_{\beta\to\infty}\langle\sigma_i\rangle^{\beta,\mathbf{h},-} = -1$. For this, it will be convenient to consider the Hamiltonian of the Ising model with – b.c. given by

$$H_\Lambda^-(\sigma) = -\sum_{\langle i,j\rangle}\frac{J}{2}(\sigma_i\sigma_j - 1) - \sum_{i\in\Lambda} h_i(\sigma_i + 1). \tag{6}$$

Note that this normalization does not change the measure $\mu_\Lambda^{\beta,\mathbf{h},-}$.

We will identify $\mathbb{Z}^2$ with the subset of $\mathbb{R}^2$ of integer coordinates. Fix a finite set $\Lambda\subset\mathbb{Z}^2$, to each site i of Λ we associate the dual plaquette $p^*(i)$ having i at its center. We call *plaquettes* the unit squares in $\mathbb{R}^2$ whose corners are in $\mathbb{Z}^{2^*}$. The *dual set* Λ^* of Λ is the subset of $\mathbb{Z}^{2^*}$ of the corners of $p^*(i)$, where i is some site of Λ.

For each configuration σ_Λ satisfying the – b.c. we will associate a family of contours as follows: for each pair of nearest neighbors sites in σ_Λ, where we have opposite signs, we consider the unit segment e joining the two sites. The dual segment e^* will be the unit segment orthogonal to e passing through the middle point of e, and joining the two sites of Λ^* that are closest to that middle point.

The union of these dual unit segments, called *edges*, will form closed curves in $\mathbb{R}^2$ such that the sites in Λ^* will have degree 0, 2 or 4 and, using some rule we can "cut" the corner when the degree is 4, see [17] for details.

This process give us a bijection between the configurations satisfying – b.c. and the finite sets of *compatible* contours in Λ^*, that is, closed self-avoiding and pairwise mutually avoiding contours in Λ^*.

Fixed the – b.c., for each contour γ there is an unique configuration σ_γ which has γ as unique contour. We define the *interior of* γ, $\operatorname{int}\gamma$, as the set of all $i\in\mathbb{Z}^2$ such that, for σ_γ, we have $\sigma_i = +1$ and $d(i,\gamma) > 1$, where d denote the Euclidean distance in $\mathbb{R}^2$. We will use the notation $\overline{\operatorname{int}\gamma}$ for the set of all $i\in\mathbb{Z}^2$ for which $\sigma_i = +1$ in σ_γ. The volume of γ is the cardinality of $\overline{\operatorname{int}\gamma}$, $\operatorname{vol}\gamma = |\overline{\operatorname{int}\gamma}|$.

Let σ_Λ be a configuration satisfying the – b.c. We have already seen that there is a finite set of contours associated to σ_Λ. For each contour γ all spins at $\overline{\operatorname{int}\gamma}\setminus\operatorname{int}\gamma$ have the same value. We say that γ is of type + (resp. type −), if the value of these spins are +1 (resp. −1).

We define over the set of signed contours a function ξ given by

$$\xi(\gamma) = \begin{cases}\exp\left(-2\beta J|\gamma| - 2\beta\sum_{i\in\overline{\operatorname{int}\gamma}} h_i\right), & \gamma \text{ of type } -;\\ \exp\left(-2\beta J|\gamma| + 2\beta\sum_{i\in\overline{\operatorname{int}\gamma}} h_i\right), & \gamma \text{ of type } +,\end{cases} \tag{7}$$

where $|\gamma|$ denote the number of unit segments that compose the contour.

Denoting by Z_Λ^- the partition function corresponding to the – b.c., it follows from a straightforward computation that:

$$Z_\Lambda^- = 1 + \sum_{n\ge 1}\frac{1}{n!}\sum_{\substack{(\gamma_1,\dots,\gamma_n)\\ \Lambda^*\text{-compatibles}}}\prod_{k=1}^{n}\xi(\gamma_k). \tag{8}$$

We remark that the value $\xi(\gamma)$ depends explicitly on the type of the contour, which is different from the usual cluster expansion. Since

$$\langle\sigma_i\rangle_\Lambda^{\beta,\mathbf{h},-} = 2\,\mu_\Lambda^{\beta,\mathbf{h},-}(\{\sigma_i = +1\}) - 1 \tag{9}$$

if we prove that $\mu_\Lambda^{\beta,\mathbf{h},-}(\{\sigma_i = +1\}) \to 0$, uniformly in Λ when $\beta\to\infty$, we are done.

Whenever $\sigma_i = +1$ there exists a contour type $+$, denoted by $\overset{+}{\gamma}$, that involves the site i. We use the notation $\overset{+}{\gamma} \odot i$ to indicate that $\overset{+}{\gamma}$ involves the site i. We say that γ involves γ', and we write $\gamma \odot \gamma'$, if any site $i \in \mathbb{Z}^d$ involved by γ', it is also involved by γ.

Before continuing the theorem's proof we need the following lemma:

Lemma 2 *Let* $\{\gamma_1, \ldots, \gamma_n\}$ *be a collection of* Λ^**-compatible signed contours. If* $\mathbf{h} \in \ell^1(\mathbb{Z}^d)$*, then*

$$e^{-2\beta\|h\|_1}\left(\prod_{k=1}^{n} e^{-2\beta J|\gamma_k|}\right) \le \prod_{k=1}^{n}\xi(\gamma_k) \le e^{2\beta\|h\|_1}\left(\prod_{k=1}^{n} e^{-2\beta J|\gamma_k|}\right).$$

Proof For a fixed collection of Λ^*-compatible signed contours $\{\gamma_1, \ldots, \gamma_n\}$ the relation involving $\odot$ determines naturally a partial order in this set. Let $\{\gamma_{r_1}, \ldots, \gamma_{r_k}\}$ be the set of all maximal elements with respect to this partial order. So we have that

$$\prod_{k=1}^{n}\xi(\gamma_k) = \prod_{l=1}^{k}\xi(\gamma_{r_l})\left(\prod_{\{j:\gamma_{r_l}\odot\gamma_j\}}\xi(\gamma_j)\right), \tag{10}$$

where the product over an empty set is equal to one. If $\{j : \gamma_{r_l} \odot \gamma_j\} = \emptyset$ for all $1 \le l \le k$, the upper and lower bounds claimed in the lemma are straightforward since $\overline{\operatorname{int}\gamma_{r_l}} \cap \overline{\operatorname{int}\gamma_{r_m}} = \emptyset$ whenever $1 \le m < l \le k$. On the other hand, for each $1 \le l \le k$ such that the set $\{j : \gamma_{r_l} \odot \gamma_j\}$ is not empty, consider the graph $G_l = (V_l, E_l)$, where the vertex set $V_l = \{\gamma_{r_l}\} \cup \{\gamma_j : \gamma_{r_l} \odot \gamma_j\}$ and the edges set E_l is the collection of all unordered pairs $\{\gamma_j, \gamma_l\}$ such that if $\gamma_m \odot \gamma_j$ or $\gamma_m \odot \gamma_l$, then $\gamma_m \odot \gamma_j$ and $\gamma_m \odot \gamma_l$, i.e., there are no intermediate contours between γ_j and γ_l.

From the definition of the contours and the edges set it is easy to see that G_l is a rooted tree where the root is the most exterior contour, i.e. this contour is not in the interior of any other contour. For $\gamma_j, \gamma_s \in V_l$, we denote by $d(\gamma_j, \gamma_s)$ the usual graph distance in G_l. Let TV_l be the set of vertices in V_l that are in the last generation of the tree, i.e., the vertices with degree one on G_l and that the distance to the root is maximal. Consider the following partition of

$$TV_l = \bigcup_{j=1}^{t(l)} TV_l(j),$$

where $t(l)$ is the cardinality of the set $\{\gamma \in V_l : d(\gamma, TV_l) = 1\}$ and for each $1 \le j \le t(l)$, the sets $TV_l(j) = \{\gamma_{j1}, \ldots, \gamma_{jr_j(l)}\}$ are the maximal subsets of TV_l possessing the same parent $\gamma_{P_l(j)}$. By rearranging, the product (10) can be written as

$$\prod_{l=1}^{k}\left[\prod_{j=1}^{t(l)}\left(\xi(\gamma_{P_j(l)})\prod_{\gamma\in TV_l(j)}\xi(\gamma)\right)\prod_{\gamma\in\tilde{G}_l}\xi(\gamma)\right], \tag{11}$$

where $\tilde{G}_l$ is the tree subgraph of G_l induced by the vertices

$$V_l \backslash (TV_l \cup \{\gamma_{P_1(l)}, \ldots, \gamma_{P_{t(l)}(l)}\}).$$

Observe that the value of the function ξ at the root γ_{r_l} appears in the product $\prod_{\gamma\in\tilde{G}_l}\xi(\gamma)$ when the graph $\tilde{G}_l$ is not empty. Putting

$$S_l(j)=\overline{\operatorname{int}\gamma_{P_j(l)}}\setminus\bigcup_{\gamma\in TV_j(l)}\overline{\operatorname{int}\gamma},$$

it follows from the definition of ξ, independent of the sign of the contour $\gamma_{P_j(l)}$, that the following bounds hold:

$$e^{-2\beta\sum_{i\in S_l(j)}h_i}\left(\prod_{\{\gamma_k:\gamma_k\in TV_l(j)\cup\{\gamma_{P_j(l)}\}\}}e^{-2\beta J|\gamma_k|}\right)\le\xi(\gamma_{P_j(l)})\prod_{\gamma\in TV_l(j)}\xi(\gamma),$$

and

$$\xi(\gamma_{P_j(l)})\prod_{\gamma\in TV_l(j)}\xi(\gamma)\le e^{+2\beta\sum_{i\in S_l(j)}h_i}\left(\prod_{\{\gamma_k:\gamma_k\in TV_l(j)\cup\{\gamma_{P_j(l)}\}\}}e^{-2\beta J|\gamma_k|}\right).$$

Proceeding as above, for each $\tilde{G}_l$, we get new $S_l(j)$'s that are disjoint for the ones previously defined. So using the reasoning iteratively we finish the proof. The proof ends when each of $\tilde{G}_l$ is only the root γ_{r_l} or the root γ_{r_l} and the elements connected to it. □

To finish the proof of the theorem we use the lemma above and, by the contour representation, we get the following upper bounds for $\mu_\Lambda^{\beta,\mathbf{h},-}(\{\sigma_i=+1\})$:

$$\begin{aligned}\mu_\Lambda^{\beta,\mathbf{h},-}(\{\sigma_i=+1\})&\le\mu_\Lambda^{\beta,\mathbf{h},-}(\{\exists\ \overset{+}{\gamma}\odot i\})\\&\le\frac{\sum_{\overset{+}{\gamma}\odot i}\xi(\overset{+}{\gamma})\big(1+\sum_{n\ge1}\sum_{\substack{\{\gamma_1,\dots,\gamma_n\}\cap\{\overset{+}{\gamma}\}=\emptyset\\ \Lambda^*\text{-compatibles}}}\prod_{k=1}^n\xi(\gamma_k)\big)}{1+\sum_{n\ge1}\sum_{\substack{\{\gamma_1,\dots,\gamma_n\}\\ \Lambda^*\text{-compatibles}}}\prod_{k=1}^n\xi(\gamma_k)}\\&\le\frac{\sum_{\overset{+}{\gamma}\odot i}\xi(\overset{+}{\gamma})\big(e^{2\beta\|\mathbf{h}\|_1}+\sum_{n\ge1}\sum_{\substack{\{\gamma_1,\dots,\gamma_n\}\cap\{\overset{+}{\gamma}\}=\emptyset\\ \Lambda^*\text{-compatibles}}}e^{2\beta\|\mathbf{h}\|_1}(\prod_{k=1}^n e^{-2\beta J|\gamma_k|})\big)}{e^{-2\beta\|\mathbf{h}\|_1}+\sum_{n\ge1}\sum_{\substack{\{\gamma_1,\dots,\gamma_n\}\\ \Lambda^*\text{-compatibles}}}e^{-2\beta\|\mathbf{h}\|_1}\big(\prod_{k=1}^n e^{-2\beta J|\gamma_k|}\big)}\\&\le\sum_{\overset{+}{\gamma}\odot i}\xi(\overset{+}{\gamma})e^{4\beta\|\mathbf{h}\|_1}\le\sum_{\gamma\odot i}\exp\left(-2\beta J|\gamma|\ +\ 2\beta\sum_{i\in\overline{\operatorname{int}\gamma}}h_i+4\beta\|\mathbf{h}\|_1\right).\end{aligned}\tag{12}$$

Using the last inequality and $|\gamma|\ge4$, we obtain

$$\mu_\Lambda^{\beta,\mathbf{h},-}(\{\sigma_i=+1\})\le\sum_{n\ge4}\exp\left(-2\beta(Jn-3\|\mathbf{h}\|_1)\right)n3^n.\tag{13}$$

Taking J such that $J>3\|\mathbf{h}\|_1$, we conclude that $\mu_\Lambda^{\beta,\mathbf{h},-}(\{\sigma_i=+1\})\to0$ when $\beta\to\infty$, uniformly in Λ.

The case $\mathbf{h}\le0$ ($h_i\le0,\forall\ i\in\mathbb{Z}^2$) with $\mathbf{h}\in\ell_1(\mathbb{Z}^2)$ follows from the first case and the FKG inequality. For an arbitrary field $\mathbf{h}$ in $\in\ell_1(\mathbb{Z}^2)$ we use FKG and reduce the problem to one of the previous cases. □

Corollary *For all the magnetic field* $\mathbf{h} \in \ell_1(\mathbb{Z}^d)$, *the Ising model presents a first-order phase transition.*

Proof First replace the fields h_i by zero at all sites of a finite large enough region Γ, in order to have $J > 3\|\tilde{\mathbf{h}}\|_1$, where $\tilde{\mathbf{h}}$ denotes the modified magnetic field. Thus there are two different measures μ_- and μ_+, by Theorem 1. Now, take the local function $A(\sigma) = \sum_{i\in\Gamma} h_i\sigma_i$. For any local function $f : \Omega \to \mathbb{R}$ we have that

$$\mu_-(fe^{\beta A}) := \int_\Omega f(\sigma)e^{\beta A(\sigma)}d\mu_- = \lim_{\Lambda\nearrow\mathbb{Z}^d} \frac{\langle f\rangle_\Lambda^{\beta,\mathbf{h},-}}{\langle e^{-\beta\sum_{i\in\Gamma} h_i\sigma_i}\rangle_\Lambda^{\beta,\mathbf{h},-}} = \frac{\langle f\rangle^{\beta,\mathbf{h},-}}{\langle e^{-\beta\sum_{i\in\Gamma} h_i\sigma_i}\rangle^{\beta,\mathbf{h},-}}.$$

Analogously for the $+$ boundary condition.

Suppose by absurd that the model with the magnetic field $\mathbf{h}$ does not present a first-order phase transition, thus for any $\beta > 0$ fixed and for all local function f we have $\langle f\rangle^{\beta,\mathbf{h},-} = \langle f\rangle^{\beta,\mathbf{h},+}$. Applying this equality for $f(\sigma) = e^{-\beta A(\sigma)}$, we get

$$\langle e^{-\beta\sum_{i\in\Gamma} h_i\sigma_i}\rangle^{\beta,\mathbf{h},-} = \langle e^{-\beta\sum_{i\in\Gamma} h_i\sigma_i}\rangle^{\beta,\mathbf{h},+}.$$

So it follows from the above equalities that $\mu_-(fe^A) = \mu_+(fe^A)$ for any $\beta > 0$ and for any local function f, which is in contradiction with Theorem 1 by taking $\beta > 0$ sufficiently large and $f(\sigma) = \sigma_i e^{-A(\sigma)}$. □

Remark In the above argument we show how we can work with contours in a non-symmetric set up and also illustrate that a finite-energy, (quasi-)local change in a infinite system does not change global properties, such as (non)uniqueness of the Gibbs state. In fact, the summability guarantees that this proof works but, it is not a necessary condition for the phenomenon. For the Semi-Infinite Ising Model on $\mathbb{Z} \times \mathbb{Z}_+$ Fröhlich and Pfister [4] showed that there are two Gibbs states at sufficiently low temperature with a constant magnetic field h only over the sites in the boundary of the lattice and, as we already mentioned before, Basuev [1] obtained a phase transition in a more interesting case, where the magnetic field is constant at all sites of the lattice $\mathbb{Z}^2 \times \mathbb{Z}_+$.

4 Absence of Phase Transition

In this section we show the absence of phase transition in the non-summable case under condition that the magnetic field satisfies $\liminf h_i > 0$ for positive fields and $\limsup h_i < 0$ for negative ones. Although the proofs of Lemma 3 and Theorem 4 below are presented for the hypercubic lattice, they are the same for any quasi-transitive connected amenable graph with uniformly bounded degree due Theorem 5 of [7]. For the proof of Theorem 4, in this section, we follow close [3].

Lemma 3 *There is no phase transition in ferromagnetic Ising models with uniform nonzero external magnetic field outside of a finite volume.*

Proof Let $\mathbf{h}$ the magnetic field and $\Lambda_0 \subset \mathbb{Z}^d$ a finite set such that, $h_i = h$ for all $i \notin \Lambda_0$ and $h \in \mathbb{R}\backslash\{0\}$. To show absence of phase transition for this model it is enough to show that $\langle\sigma_i\rangle^{\beta,\mathbf{h},-} = \langle\sigma_i\rangle^{\beta,\mathbf{h},+}$ for all $\beta > 0$ and $i \in \mathbb{Z}^d$.

Consider $\Lambda_0 = \{k\} \subseteq \Lambda \subset \mathbb{Z}^d$, from now $\langle\ .\ \rangle_\Lambda^{\beta,h,\omega}$ and $Z_\Lambda^{\beta,h,\omega}$ denote respectively the expected value and the partition function with respect to the Gibbs measure defined by the

Hamiltonian (1) with boundary condition ω and constant magnetic field h. It follows from the definition that, for all $i \in \Lambda$

$$\langle \sigma_i \rangle_\Lambda^{\beta,\mathbf{h},\omega} = \langle \sigma_i \cdot e^{\beta(h_k - h)\sigma_k} \rangle_\Lambda^{\beta,h,\omega} \frac{Z_\Lambda^{\beta,h,\omega}}{Z_\Lambda^{\beta,\mathbf{h},\omega}}. \tag{14}$$

We know that the expected value $\langle \sigma_i \cdot e^{\beta(h_k - h)\sigma_k} \rangle_\Lambda^{\beta,h,\omega}$ is independent of the boundary conditions in the thermodynamical limit. So, we need to show that $Z_\Lambda^{\beta,h,\omega} / Z_\Lambda^{\beta,\mathbf{h},\omega}$ it is also independent of the boundary conditions when $\Lambda \nearrow \mathbb{Z}^d$.

In order to evaluate the limit of the above ratio we will need to consider boundary conditions in $\Lambda \backslash \{k\}$. We define ω_1 as $(\omega_1)_i = \omega_i$ for all $i \in \partial\Lambda$ with $(\omega_1)_k = +1$ and, ω_2 by $(\omega_2)_i = \omega_i$ for all $i \in \partial\Lambda$ and $(\omega_2)_k = -1$ so

$$\frac{Z_\Lambda^{\beta,h,\omega}}{Z_\Lambda^{\beta,\mathbf{h},\omega}} = \frac{e^{\beta h} \cdot Z_{\Lambda\backslash\{k\}}^{\beta,h,\omega_1} + e^{-\beta h} \cdot Z_{\Lambda\backslash\{k\}}^{\beta,h,\omega_2}}{e^{\beta h_k} \cdot Z_{\Lambda\backslash\{k\}}^{\beta,h,\omega_1} + e^{-\beta h_k} \cdot Z_{\Lambda\backslash\{k\}}^{\beta,h,\omega_2}}. \tag{15}$$

To show that the above expression in the thermodynamic limit is independent of the boundary condition ω, it is enough to show that $Z_{\Lambda\backslash\{k\}}^{\beta,h,\omega_2} / Z_{\Lambda\backslash\{k\}}^{\beta,h,\omega_1}$ does not depend on ω_1 and ω_2. To do this we take $i = k$ in (14) and $h_k \to +\infty$, and we obtain

$$\begin{aligned} 1 &= \mu_\Lambda^{\beta,h,\omega}(\{\sigma_k = +1\}) \lim_{h_k \to \infty} \left(e^{\beta(h_k - h)} \frac{e^{\beta h} \cdot Z_{\Lambda\backslash\{k\}}^{\beta,h,\omega_1} + e^{-\beta h} \cdot Z_{\Lambda\backslash\{k\}}^{\beta,h,\omega_2}}{e^{\beta h_k} \cdot Z_{\Lambda\backslash\{k\}}^{\beta,h,\omega_1} + e^{-\beta h_k} \cdot Z_{\Lambda\backslash\{k\}}^{\beta,h,\omega_2}} \right) \\ &= \mu_\Lambda^{\beta,h,\omega}(\{\sigma_k = +1\}) \left(1 + e^{-2\beta h} \cdot \frac{Z_{\Lambda\backslash\{k\}}^{\beta,h,\omega_2}}{Z_{\Lambda\backslash\{k\}}^{\beta,h,\omega_1}} \right) \end{aligned}$$

for all finite set Λ containing k. Then,

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{Z_{\Lambda\backslash\{k\}}^{\beta,h,\omega_2}}{Z_{\Lambda\backslash\{k\}}^{\beta,h,\omega_1}} = e^{2\beta h} \frac{\mu^{\beta,h,\omega}(\{\sigma_k = -1\})}{\mu^{\beta,h,\omega}(\{\sigma_k = +1\})} \tag{16}$$

and the argument follows by induction in $|\Lambda_0|$. □

Theorem 4 *If* $\mathbf{h} \in \ell_\infty(\mathbb{Z}^d)$ *is such that* $\liminf_{i \in \mathbb{Z}^d} h_i > 0$, *then the Ising model with external field* $\mathbf{h}$ *has no phase transition.*

Proof We omit the parameter β since the argument is valid for all $\beta > 0$. Let be $\overline{h} = \sup_{i \in \mathbb{Z}^d} h_i$, $\underline{h} = \liminf_{i \in \mathbb{Z}^d} h_i$ and ε a positive number such that $0 < \varepsilon < \underline{h}/2$. We denote by $\Lambda \subset \mathbb{Z}^d$ a finite subset containing $\{i \in \mathbb{Z}^d; |h_i| < \varepsilon\}$, h_Λ the restriction of the external field $\mathbf{h}$ to the volume Λ, $\overline{h}_\Lambda$ the constant magnetic field in the set Λ with the value $\overline{h}$, analogously for $\underline{h}$ and $\underline{h}_\Lambda^\varepsilon$ the constant magnetic field taking values $\underline{h} - \varepsilon$. It follows from Fundamental Theorem of Calculus that for any finite $\Gamma \subset \mathbb{Z}^d$ with $\Lambda \subset \Gamma$, we have that the difference

$$\langle \sigma_i \rangle_\Gamma^{h_\Lambda, \overline{h}_{\Gamma\backslash\Lambda}, +} - \langle \sigma_i \rangle_\Gamma^{h_\Lambda, \underline{h}_{\Gamma\backslash\Lambda}^\varepsilon, -}$$

is equal to

$$\beta \int_{\underline{h}-\varepsilon}^{\overline{h}} \sum_{j \in \Gamma\backslash\Lambda} \langle \sigma_i ; \sigma_j \rangle_\Gamma^{h_\Lambda, x_{\Gamma\backslash\Lambda}, +} dx + \langle \sigma_i \rangle_\Gamma^{h_\Lambda, \underline{h}_{\Gamma\backslash\Lambda}^\varepsilon, +} - \langle \sigma_i \rangle_\Gamma^{h_\Lambda, \underline{h}_{\Gamma\backslash\Lambda}^\varepsilon, -}.$$

By Lemma 3 we know that there is no phase transition for the ferromagnetic models with uniform nonzero magnetic field outside of a finite volume. Then, by the FKG inequality taking the limit when $\Gamma \nearrow \mathbb{Z}^d$ we have:

$$0 \le \langle \sigma_i \rangle^{h_\Lambda, \overline{h}_{Z^d \setminus \Lambda}, +} - \langle \sigma_i \rangle^{h_\Lambda, \underline{h}^\varepsilon_{Z^d \setminus \Lambda}, -} = \lim_{\Gamma \nearrow \mathbb{Z}^d} \beta \int_{\underline{h}-\varepsilon}^{\overline{h}} \sum_{j \in \Gamma \setminus \Lambda} \langle \sigma_i ; \sigma_j \rangle_\Gamma^{h_\Lambda, x_{\Gamma \setminus \Lambda}, +} \, dx.$$

Now, taking the limit when $\Lambda \nearrow \mathbb{Z}^d$ we obtain:

$$0 \le \langle \sigma_i \rangle^{\mathbf{h}, +} - \langle \sigma_i \rangle^{\mathbf{h}, -} \le \lim_{\Lambda \nearrow \mathbb{Z}^d} \lim_{\Gamma \nearrow \mathbb{Z}^d} \beta \int_{\underline{h}-\varepsilon}^{\overline{h}} \sum_{j \in \Gamma \setminus \Lambda} \langle \sigma_i ; \sigma_j \rangle_\Gamma^{h_\Lambda, x_{\Gamma \setminus \Lambda}, +} \, dx.$$

Since the truncated correlation functions are non-increasing in h_k $(k \in \mathbb{Z}^d)$, we get

$$\begin{aligned} 0 \le \langle \sigma_i \rangle^{\mathbf{h}, +} - \langle \sigma_i \rangle^{\mathbf{h}, -} &\le \lim_{\Lambda \nearrow \mathbb{Z}^d} \lim_{\Gamma \nearrow \mathbb{Z}^d} \beta \int_{\underline{h}-\varepsilon}^{\overline{h}} \sum_{j \in \Gamma \setminus \Lambda} \langle \sigma_i ; \sigma_j \rangle_\Gamma^{\underline{h}^\varepsilon_\Lambda, x_{\Gamma \setminus \Lambda}, +} \, dx \\ &= \lim_{\Lambda \nearrow \mathbb{Z}^d} \lim_{\Gamma \nearrow \mathbb{Z}^d} \langle \sigma_i \rangle_\Gamma^{\underline{h}^\varepsilon_\Lambda, \overline{h}_{\Gamma \setminus \Lambda}, +} - \langle \sigma_i \rangle_\Gamma^{\underline{h}^\varepsilon_\Lambda, \underline{h}^\varepsilon_{\Gamma \setminus \Lambda}, +} \\ &\le \lim_{\Lambda \nearrow \mathbb{Z}^d} \langle \sigma_i \rangle_\Lambda^{\underline{h}^\varepsilon_\Lambda, +} - \langle \sigma_i \rangle^{\underline{h}-\varepsilon, +} \\ &= 0. \end{aligned}$$

The last inequality comes again from FKG. By standard arguments [8], there is only one Gibbs state for the model. The analogous result holds when the magnetic field is negative. □

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